

Dimensional Dependence of Critical Exponent of the Anderson Transition in the Orthogonal Universality Class

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We report a numerical study of the Anderson transition in the $d = 4$ and $d = 5$ dimensional Anderson model of localization. We have used the transfer matrix method and a finite size scaling analysis to estimate the critical exponent ν . We find $\nu = 1.156 \pm 0.014$ for $d = 4$ and $\nu = 0.969 \pm 0.015$ for $d = 5$. We also report a new Borel-Padé analysis of existing ϵ expansion results that incorporates the correct asymptotic behavior for infinite dimension and gives better agreement with available numerical results for dimensions $d = 3, 4, 5$ and 6 .

KEYWORDS: Anderson transition, transfer matrix, finite size scaling, kicked rotor model, nonlinear σ model, ϵ expansion, Borel-Padé analysis

1. Introduction

The classical motion of an electron in a random potential is diffusive. Anderson¹⁾ realized that diffusion may be suppressed in the corresponding quantum problem. This phenomenon was subsequently called Anderson localisation. The suppression of diffusion is associated with a metal-insulator transition called the Anderson transition. It was eventually realised the Anderson transition is a continuous quantum phase transition and that the renormalization group, developed to describe continuous thermal phase transitions, is applicable.²⁾

One feature of continuous phase transitions is the power law dependence of various physical quantities, called critical phenomena, observed near the transition.³⁾ The exponents appearing in these laws are expected to be universal depending only on the symmetries of the Hamiltonian and the dimensionality d of the system. Two important symmetries for the Anderson transition are time reversal symmetry and spin rotation symmetry (see Ref. 4 for a full discussion of relevant symmetries and universality classes).

In this paper we are concerned with systems which have both time reversal symmetry and spin rotation symmetries. Such systems are said to have orthogonal symmetry. For systems with this symmetry the lower critical dimension is $d = 2$. In $d = 1$ and $d = 2$, arbitrarily weak randomness is enough to cause localisation. While, in $d > 2$, localization only occurs when the randomness or disorder is sufficiently strong. Our focus is on the dimensionality dependence of the critical exponent ν that describes the power law divergence of the correlation length around the critical point.

In common with other continuous phase transitions the Anderson transition is described by a field theory.⁵⁻⁷⁾ One of the results of the field theoretical approach was the development of a perturbation series, in powers of $\epsilon = d - 2$, for the critical exponent. In Ref. 8, this calculation was carried to five-loop order. Unfortunately the series is very poorly convergent. To obtain estimates for the critical exponent a Borel-Padé analysis is required. However, the analysis presented in Ref. 8 gives values for the critical exponent that are in clear disagreement with the available numerical results and also violate the lower bound^{9, 10)} $\nu \geq 2/d$ for the critical exponent. Thus, it seemed that the field theoretic approach could provide only a quali-

tative description of the transition in $d \geq 3$. One aim of this paper is to show that a revised Borel-Padé analysis that takes into account the correct asymptotic behaviour of the critical exponent in the limit $d \rightarrow \infty$ gives much better agreement with numerical results, at least as good or better than any other approach.

Another motivation for our work is that experimental realisations of higher dimensional Anderson transitions are now foreseeable. Such realisations will probably not involve electrons, where, in any case, there has been only very slow progress in understanding how the Coulomb interaction between electrons affects the transition.¹¹⁻¹³⁾ Rather, they may be possible in systems that display dynamical localisation.

Dynamical localisation is the analogue of Anderson localisation in momentum space for periodically driven systems. One such a system is the one dimensional quantum kicked rotor, which was experimentally realised by Moore et al.¹⁴⁾ In Ref. 15, a mapping between a one dimensional kicked rotor where the amplitude of the periodic kick is modulated in an aperiodic manner controlled by three incommensurate frequencies and a three dimensional lattice model was derived. A realisation of this system in a cold atomic gas was reported in Ref. 16 where a transition driven by the kicking strength between a phase of diffusion and localisation in momentum space was observed. A scaling analysis of the experimental data yielded a value $\nu = 1.4 \pm .3$ for the critical exponent. Numerical simulations of the experimental system, reported in the same reference, gave $\nu = 1.60 \pm .05$. In subsequent work the precision of the experimental estimate of the critical exponent has improved considerably with a value $\nu = 1.63 \pm .05$ reported in Ref. 17. These values are in agreement with the most precise numerical estimates $\nu = 1.571 \pm .004$ ^{18, 19)} and $\nu = 1.590 \pm .006$ ^{20, 21)} of the critical exponent for the $d = 3$ orthogonal universality class of the Anderson transition. While the mapping in Ref. 15 with the Anderson model is not exact, since the lattice model obtained has a quasi-periodic rather than random potential, there is clearly strong evidence that the transitions are in the same universality class. Moreover, the mapping makes clear that the dimensionality of the lattice model is determined by the number of incommensurate frequencies. Higher dimensional Anderson transitions could be

realised in the same rotor but subject to modulation by appropriate numbers of incommensurate frequencies. Therefore, a higher precision estimation of some of the corresponding critical exponents seems worthwhile.

The paper is organized as follows. In Section 2 we describe the numerical estimation of the critical exponent for $d = 4$ and $d = 5$. In Section 3 we describe our new Borel-Padé analysis. In Section 4 we compare the values obtained with our Borel-Padé analysis with available numerical results and various analytical approaches. In Section 5 we conclude.

2. Numerical estimation of critical exponent ν

2.1 The Anderson model of localisation

The Hamiltonian of the Anderson model of localisation is

$$H = \sum_{\mathbf{r}} \epsilon_{\mathbf{r}} |\mathbf{r}\rangle \langle \mathbf{r}| + \sum_{\mathbf{r}, \mathbf{r}'} V_{\mathbf{r}\mathbf{r}'} |\mathbf{r}\rangle \langle \mathbf{r}'|, \quad (1)$$

$$V_{\mathbf{r}\mathbf{r}'} = \begin{cases} 1 & (\mathbf{r}, \mathbf{r}' \text{ are nearest neighbors}) \\ 0 & (\text{otherwise}) \end{cases}. \quad (2)$$

Here, \mathbf{r}, \mathbf{r}' are lattice points on d -dimensional cubic lattice. The unit of energy has been set equal to the hopping energy. The site energies $\epsilon_{\mathbf{r}}$ are independently and identically distributed according to the following probability density function,

$$p(\epsilon_{\mathbf{r}}) = \begin{cases} 1/W & (|\epsilon_{\mathbf{r}}| \leq W/2) \\ 0 & (\text{otherwise}) \end{cases}. \quad (3)$$

The choice of the probability density function does not change the universality class.^{18,19)} The degree of disorder is set by the parameter W . We used MT2203 of the Intel MKL library to generate the required random numbers.

2.2 Transfer matrix method

The numerical method we have used is described in detail in Ref. 19. Here, we give only a very brief description and refer the reader to that reference for details.

We consider a very long (length L_x) quasi-one dimensional bar of cross section L^{d-1} and study the localisation of electrons on this bar using the transfer matrix method. The results of the transfer matrix calculation are an estimate of the smallest positive Lyapunov exponent γ and the standard deviation σ describing the statistical error in this estimate. The smallest positive Lyapunov exponent corresponds to the inverse of the localisation length for electrons on the bar. The Lyapunov exponent depends on the energy E , the disorder W , the linear cross-section L and the boundary conditions imposed in the transverse direction. We set $E = 0$ throughout, which corresponds to the band center, and we impose periodic boundary conditions in the transverse directions.

Data for $W = 30, 31, \dots, 40$, $L = 4, 6, \dots, 20$ for $d = 4$, and $W = 52, 53, \dots, 64$, $L = 4, 5, \dots, 10$ for $d = 5$ were accumulated. Almost all of data have a precision of 1% which required between the order of 10^4 and 10^5 iterations depending on disorder and dimension. To avoid round-off error QR factorizations were performed after every 4 transfer matrix multiplications. To ensure that the precision of the Lyapunov exponents was estimated correctly we set $r = 4$ in Eq. (34) of Ref. 19.

2.3 Finite size scaling analysis

To estimate the critical exponent and other quantities we analyse the data for the smallest positive Lyapunov exponent by using the method of finite size scaling to fit the disorder and size dependence of the dimensionless quantity

$$\Gamma = \gamma L. \quad (4)$$

As above we give only a brief description here and refer the reader to Ref. 19 for further details.

We found that it was not necessary to consider corrections due to irrelevant variables, instead it was sufficient to fit the data to

$$\Gamma = F(\phi(w, L)), \quad (5)$$

where

$$\phi(w, L) = u(w)L^{1/\nu}, \quad (6)$$

$$w = \frac{W - W_c}{W_c}, \quad (7)$$

and W_c is the critical disorder. The scaling function F was expanded as follows

$$F(\phi) = \sum_{j=0}^n a_j \phi^j. \quad (8)$$

Non-linearity of the relevant scaling variable was treated using the expansion

$$u(w) = \sum_{j=1}^m b_j w^j. \quad (9)$$

The integers m and n define the order of the expansions. The best fit is found by minimizing the chi-squared statistic χ^2 in the usual way. The quality of the fit is assessed using the goodness of fit probability, which is determined from the minimum value of χ^2 and the number of degrees of freedom in the fit.

For $d = 4$ we found $m = 3$ and $n = 3$ in Eqs. (8) and (9) gave an acceptable fit. The number of parameters and the number of data used in the finite size scaling analysis were respectively 8 and 99. For the best fit $\chi^2 = 91.6$ which gives a goodness of fit of 0.46. The fit is displayed in Figs. 1 and 2. The estimates of the critical exponent ν , the critical disorder W_c and the critical value Γ_c of Γ , together with the standard deviations of these estimates, are

$$\begin{aligned} \nu &= 1.156 \pm 0.014, \\ \Gamma_c &= 2.76 \pm 0.01, \\ W_c &= 34.62 \pm 0.03. \end{aligned} \quad (10)$$

For $d = 5$, $m = 1$ and $n = 1$ gave an acceptable fit. The number of parameter and the number of data are 4 and 91. For the best fit $\chi^2 = 84.0$ and the goodness of fit is 0.57. The fit is displayed in Figs. 3 and 4.

$$\begin{aligned} \nu &= 0.969 \pm 0.015, \\ \Gamma_c &= 3.41 \pm 0.01, \\ W_c &= 57.3 \pm 0.05. \end{aligned} \quad (11)$$

These results agree with numerical estimates reported in previous works^{22–24)} but are considerably more precise.

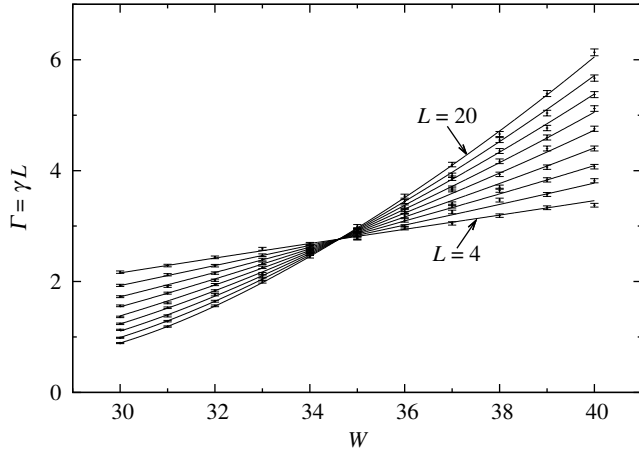


Fig. 1. The data from the transfer matrix calculation and the finite size scaling analysis for the Anderson transition in the $d = 4$ orthogonal universality class. Here the data are plotted versus disorder and different curves correspond to different system sizes. The common crossing point of different curves indicates the critical disorder separating the localised and diffusive regimes in $d = 4$. The increase in the slope at the critical disorder with system size is related to the critical exponent.

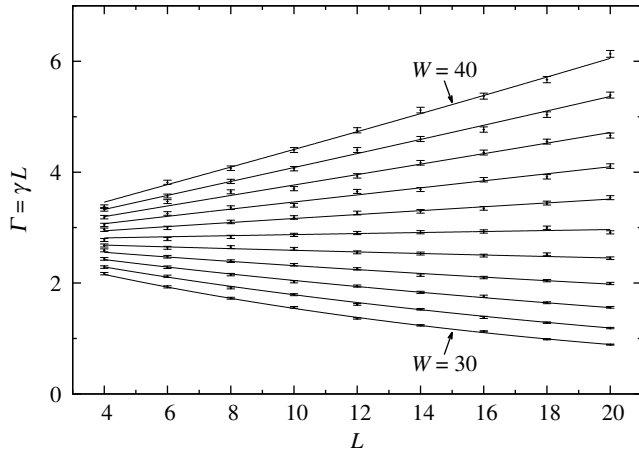


Fig. 2. Here the data for the Anderson transition in the $d = 4$ orthogonal universality class are plotted versus system size. The different curves correspond to different disorders. The change of sign of the slope indicates the transition from localised to extended states in $d = 4$.

3. The Borel-Padé analysis

In Ref. 8 a perturbation analysis carried to five-loop order yielded the following result for dimensionality dependence of the critical exponent

$$\nu = \frac{1}{\epsilon} - \frac{9}{4}\zeta(3)\epsilon^2 + \frac{27}{16}\zeta(4)\epsilon^3 + O(\epsilon^4). \quad (12)$$

Here

$$\epsilon \equiv d - 2. \quad (13)$$

Straightforward application of Borel-Padé analysis for this expression gives the following⁸⁾

$$\nu \simeq \frac{1}{\epsilon^2} \int_0^\infty dt e^{-t/\epsilon} \frac{1 + \frac{3\zeta(4)}{16\zeta(3)}t}{1 + \frac{3\zeta(4)}{16\zeta(3)}t + \frac{3}{8}\zeta(3)t^3}. \quad (14)$$

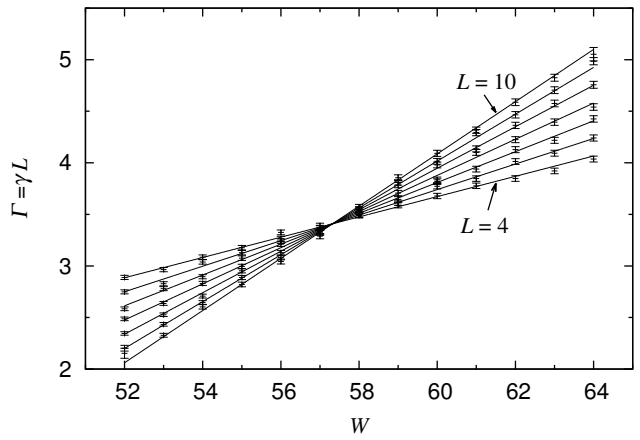


Fig. 3. The data from the transfer matrix calculation and the finite size scaling analysis for the Anderson transition in the $d = 5$ orthogonal universality class. Compared with $d = 4$ the transition occurs at a much higher disorder consistent with it becoming progressively more difficult to localize electrons in higher dimensions.

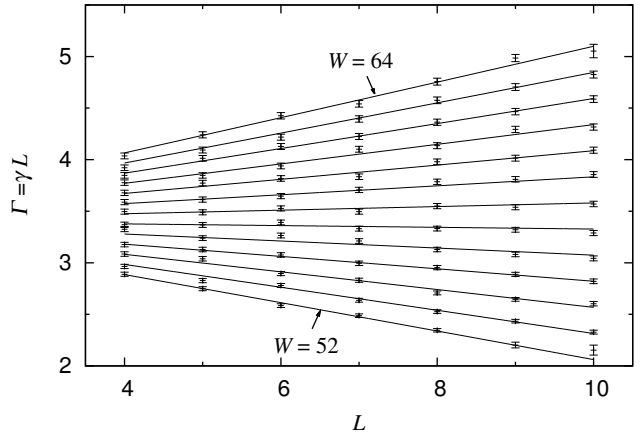


Fig. 4. Exactly similar to Fig. 2 except that here the data and fit are for the Anderson transition in the $d = 5$ orthogonal universality class.

The values for the exponents obtained from this expression for the $d = 3, 4, 5$ and 6 dimensional orthogonal universality classes are listed in Table I. There is clear discrepancy with the numerical results. As we explain below, the agreement is considerably improved if the Borel-Padé analysis is revised to take account of the correct asymptotic behaviour.

In the limits of $d \rightarrow 2^+$ and $d \rightarrow \infty$ dimensions, respectively, the asymptotic behaviours of Eq. (14) are

$$\lim_{\epsilon \rightarrow 0^+} \epsilon \nu(\epsilon) = 1, \quad (15)$$

and

$$\lim_{\epsilon \rightarrow \infty} \nu(\epsilon) = 0. \quad (16)$$

On the basis of several approaches (see the next section), it is expected that the correct asymptotic behaviour is, in fact,

$$\lim_{\epsilon \rightarrow \infty} \nu(\epsilon) = \frac{1}{2}. \quad (17)$$

To incorporate this behaviour into the Borel-Padé analysis, we

separate the term which determines the asymptotic behaviour by rewriting the ϵ -expansion as

$$\nu = \frac{1}{2} + \frac{1-\alpha}{\epsilon} + \frac{1}{\epsilon}f(\epsilon),$$

$$f(\epsilon) = \alpha - \frac{1}{2}\epsilon - \frac{9}{4}\zeta(3)\epsilon^3 + \frac{27}{16}\zeta(4)\epsilon^4 + O(\epsilon^5), \quad (18)$$

Here, α is an arbitrary real number. After this separation the Borel-Padé analysis is applied only to $f(\epsilon)$,

$$f(\epsilon) \approx \frac{1}{\epsilon} \mathcal{P} \int_0^\infty dt e^{-t/\epsilon} h(t), \quad (19)$$

where

$$h(t) = \frac{\alpha + \left(\frac{3\zeta(4)}{16\zeta(3)}\alpha - \frac{1}{2}\right)t - \left(\frac{3}{4}\zeta(3)\alpha + \frac{3\zeta(4)}{32\zeta(3)}\right)t^2}{1 + \frac{3\zeta(4)}{16\zeta(3)}t - \frac{3}{4}\zeta(3)t^2}. \quad (20)$$

Here, \mathcal{P} indicates the Cauchy principal value. From this we obtain the following expression

$$\nu \approx \frac{1}{2} + \frac{1 + \frac{\zeta(4)}{8\zeta(3)^2}}{\epsilon} - \frac{1}{3\zeta(3)\epsilon^2}g(\epsilon),$$

$$g(\epsilon) = c_+ e^{-t_+/\epsilon} \text{Ei}\left(\frac{t_+}{\epsilon}\right) + c_- e^{-t_-/\epsilon} \text{Ei}\left(\frac{t_-}{\epsilon}\right), \quad (21)$$

where, c_\pm, t_\pm are given by

$$c_\pm = 1 + \frac{3\zeta(4)^2}{64\zeta(3)^3} \pm \frac{9\zeta(4)}{\sqrt{768\zeta(3)^3 + 9\zeta(4)^2}} \left(1 + \frac{\zeta(4)^2}{64\zeta(3)^3}\right),$$

$$\approx 1.3, 0.7632.$$

$$t_\pm = \frac{3\zeta(4) \pm \sqrt{768\zeta(3)^3 + 9\zeta(4)^2}}{24\zeta(3)^2},$$

$$\approx 1.151, -0.9637. \quad (22)$$

Appearances to the contrary, the approximation does not depend on the value of the parameter α . Also, for the revised Borel-Padé analysis the asymptotic behavior in the limit of $\epsilon \rightarrow 0^+$ is unchanged, i. e. the same as that given in Eq. (15).

4. Discussion

The available analytical and numerical results for the exponents in dimensions $d = 3, 4, 5$ and 6 are listed in Table I and displayed in Fig. 5.

The self-consistent theory²⁵⁾ of Anderson localisation predicts the following dimensionality dependence of the critical exponent

$$\nu = \frac{1}{\epsilon} \quad (2 < d < 4),$$

$$\nu = \frac{1}{2} \quad (d \geq 4). \quad (23)$$

Comparison of row one of Table I with the numerical results shows immediately that, as expected, the predictions of the self-consistent theory for critical phenomena are not quantitatively accurate. Moreover, the numerical results leave no doubt that the dimensionality dependence of the exponent persists beyond $d = 4$. The prediction that the upper critical dimension is $d = 4$ is, therefore, not correct. In fact, the consensus^{24,26-28)} is that the upper critical dimension is $d = \infty$ and that $\nu = 1/2$ in that limit.

The predictions of the Borel-Padé analysis of Ref. 8 given

by Eq. (14) are in poor agreement with the numerical results. The asymptotic behaviour for $d \rightarrow \infty$ is clearly incorrect. Moreover, the estimates for higher dimensions violate the well known lower bound^{9,10)} for the exponent

$$\nu \geq \frac{2}{d}. \quad (24)$$

In Table I and Fig. 5 we also compare with the prediction

$$\nu = \frac{1}{2} + \frac{1}{\epsilon} \quad (d > 2), \quad (25)$$

of the semi-classical theory of the Anderson transition presented in Ref. 29. The agreement with the numerical results, though far from exact, is a vast improvement on either the self-consistent theory or the Borel-Padé analysis of Ref. 8.

Finally, we turn to our revised Borel-Padé analysis. Compared with the semi-classical theory, the agreement is slightly worse for $d = 3$ but better for $d = 4, 5$ and 6 .

So far we have considered only integer dimensions. Numerical results for the Anderson transition on fractals are also available.^{28,30,31)} Fractals are described by several dimension, among them the fractal dimension and the spectral dimension. Previous numerical work²⁸⁾ has established that it is the spectral dimension that it is important in determining the universality class. In Ref. 28 the following fit

$$\nu = \frac{1}{2} + \frac{\alpha}{\epsilon}, \quad \alpha \approx 0.8, \quad (26)$$

to data for several fractals combined with data for dimension $d = 3$ and 4 was reported. This formula, together with data for fractals reported in Refs. 30 and 31 is plotted in Fig. 6. Eq. (26) does not seem to have the correct asymptotic behaviour for $\epsilon \rightarrow 0^+$ unless we set $\alpha = 1$, in which case it becomes identical to the result of the semi-classical theory. Also, not all the results for fractals are consistent with this fit.

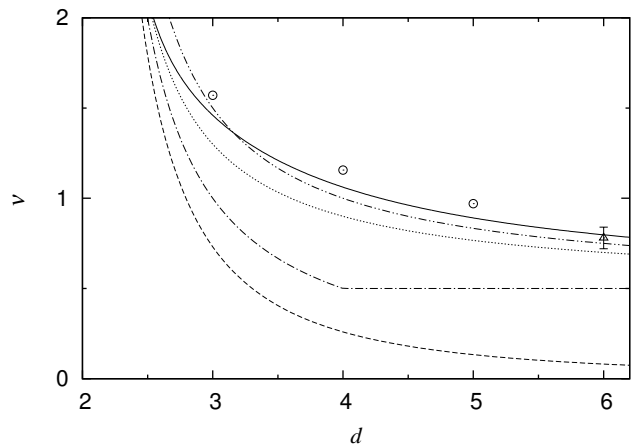


Fig. 5. The dimensionality dependence of the critical exponent ν of the Anderson transition. The points are numerical estimates in $d = 3, 4, 5$ (\circ) Ref. 19 and this work) and $d = 6$ (\triangle) Ref. 24). Error bars are standard deviations. They are omitted when the error is smaller than the symbol size. The lines are analytical predictions: our revised Borel-Padé analysis Eq. (19) (solid), the semiclassical theory Eq. (25) (dash dot dot), the fit given by Eq. (26) (dot), the self-consistent theory Eq. (23) (dash dot), and the Borel-Padé analysis of Eq. (14) (dash).

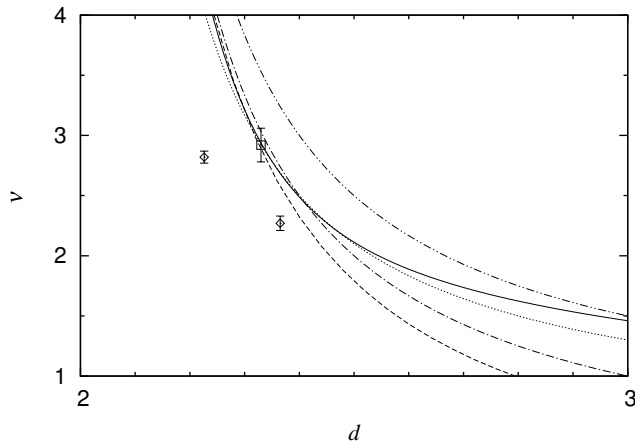


Fig. 6. The dimensionality dependence of the critical exponent ν of the Anderson transition for various fractals. The points are numerical estimates ((□) Ref. 30, (◇) Ref. 31). The lines have the same meaning as in Fig. 5.

ν	$d = 3$	$d = 4$	$d = 5$	$d = 6$
Eq. (23)	1	0.5	0.5	0.5
Eq. (14)	0.731	0.259	0.133	0.081
Eq. (25)	1.5	1.0	0.83	0.75
Eq. (19)	1.460	1.061	0.891	0.798
This study				
ref. 19	$1.571 \pm .004$	$1.156 \pm .014$	$0.969 \pm .015$	
ref. 23		$1.12 \pm .05$	$0.94 \pm .05$	
ref. 21	$1.590 \pm .06$			
ref. 24	$1.52 \pm .06$	$1.03 \pm .07$	$0.84 \pm .06$	$0.78 \pm .06$

Table I. Comparison between numerical and analytical estimates of the critical exponent ν for $d = 3, 4, 5, 6$.

5. Conclusions

We have presented more precise numerical estimates for the critical exponent of the Anderson transition in the $d = 4$ and $d = 5$ orthogonal universality classes. We have also presented a revised Borel-Padé analysis, which takes account of the theoretically expected asymptotic behaviour of the exponent as $d \rightarrow \infty$. This revised Borel-Padé analysis gives very acceptable agreement with the available numerical estimates of the exponent. More precise estimates of the critical exponent for transitions on various fractals would be useful to check the behaviour between $2 < d < 3$.

The improved estimates of the exponent also have relevance for possible future experimental realisation of the $d = 4$ and $d = 5$ orthogonal universality classes in the quantum kicked rotor system in cold atomic gases.

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